§ Quantum entropy of BHs

BHs in asymptotically $\mathbb{R}^{1,3}$, preserving 4 susys.

Attractor mechanism $\Rightarrow$ $S_{BH} = \dfrac{A_H(\vec{Q},\vec{P})}{4}$

calculable as a function of charges.

known Microscopic count $B_{2n}(\vec{Q},\vec{P})$ $\to$ states/BHs. $\mathcal{N} = 8,\ 4,\ 2$

All cases: $\log B_{2n}(\vec{Q},\vec{P}) = \dfrac{A_H(\vec{Q},\vec{P})}{4} + \dots$ as $A_H \to \infty$.

$\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$ - BPS

Best understood example

B_{14} ✓ B_6 ✓ B_2 some cases.

$\frac{1}{8}$-BPS states in $\mathcal{N}=8$ theory

Type II / T^6. $(g_{\mu\nu}, A_\mu^{A=1,2,\dots 28}, \varphi^i, \psi)$.

$$\boxed{\begin{array}{l}(Q^A, P^A) = q^a \\ A=1,\dots 28 \qquad a=1,\dots 56\end{array}}$$

U-duality $\Delta = C_{abcd}\, q^a q^b q^c q^d$ $\quad a,b = 1,2,\dots 56$.

$E_{7,7}$ invariant. $\underline{S_{BH} = \pi\sqrt{\Delta}}$

$B_{14}(\Delta)$ can be counted explicitly ~~...~~ in weakly coupled string theory. (Using U-duality, most general $\frac{1}{8}$-BPS states are represented by 4 charges (D1-D5-p-KK))

Δ	3	4	7	8	11	12	15	...
$B_{14}(\Delta)$	8	12	39	56	152	208	513	...

$\hookrightarrow$ coefficients of a certain modular form $\Rightarrow$ Analytic formula [Hardy-Ramanujan-Rademacher] expansion.

$$B_{14}(\Delta) = \sum_{c\geq 1}^{\infty} \frac{1}{c^{9/2}} K_c(\Delta)\, \tilde{I}_{7/2}\left(\frac{\pi\sqrt{\Delta}}{c}\right)$$

$$= \tilde{I}_{7/2}(\pi\sqrt{\Delta}) \left(1 + O(e^{-\pi\sqrt{\Delta}/2})\right)$$

$$\underset{\Delta\to\infty}{\approx} e^{\pi\sqrt{\Delta} - 2\log\Delta + \dots}$$

$\uparrow A_H/4$

Q: Can we do better from macroscopic side? L3 1b

$$\log B_{2n}(\vec{\alpha},\vec{\beta}) = \frac{A_H}{4}(\alpha,\vec{\beta}) + (\ldots)$$

Calculate?

Match w/ expansions of B_{2n}

eg. $B_{4n}(\Delta)$

$\Delta \to \infty$. ?

+ AdS_2 $\quad ds^2 = -r^2 dt^2 + \frac{dr^2}{r^2}$ Lorentzian

$ds^2 = r^2 dt^2 + \frac{dr^2}{r^2}$ Euclidean

Hyperbolic disk

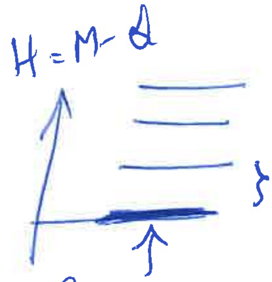
§ Quantum entropy of BHs

Classically, near-horizon $AdS_2 \times S^2$ is decoupled from rest of system.

Assume (for now) this is also true in the quantum theory ⊛

(Will revisit ⊛ in L4).

$\Rightarrow \quad Z_{AdS_2}(\vec{Q}, \vec{P}) = Z_{CFT_1}(\vec{Q}, \vec{P})$.



CFT$_1$: Assumption ⊛ $\Rightarrow$

$\Rightarrow H = 0$ on ground states

$\Rightarrow Z_{CFT_1} = \mathrm{Tr}_{\mathcal{H}}\, e^{-\beta H} = \mathrm{Tr}\, 1$

$= \dim \mathcal{H}_{BH}$. ~~[illegible]~~

→ Ignoring $(-1)^F$ for now, will revisit.

$Z_{AdS_2} = ?$

$$Z_{AdS_2} = \int_{\substack{AdS_2 \\ \text{boundary} \\ \text{conditions}}} Dg_{\mu\nu}\, DA_\mu^i\, D\varphi^a\, D\psi \times \exp(-S_{grav}(g_{\mu\nu}, A, \varphi, \psi))$$

Set-up : $S_{grav} = S_{grav}^{bulk} + S_{grav}^{bdry}$

Gibbons-Hawking-York

$g_{\mu\nu}$ $\quad \int \sqrt{g}\, R + \int \sqrt{g}|_{bdry}\, K + \ldots$

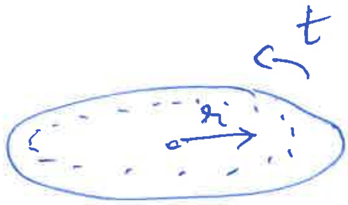
A_μ $\quad \int \sqrt{g}\, F^2 + \int \sqrt{g} \oint A^\mu F_{\mu r} + \ldots$

Note: (Always $\vec{P}$ fixed)

B.C. fix $\vec{Q}$

Microcanonical

Why?



AdS$_2$

$A_t^i = Q^i r + a_0$ $\quad (F^i_{tr} = Q^i)$

Dominates $\Rightarrow$ Fix Q^i.

(In higher dimensional systems, potential mode dominates & is fixed).

Saddle-pt. evaluation of $Z_{AdS_2} \Rightarrow$ classical EOM

$$\Rightarrow Z_{AdS_2} = e^{S^0_{BH}(\vec{q},\vec{p})} + \dots$$

classical Attractor

Origin?

1. Higher-dimension operators in UV
2. Logs of massless fields
3. Other gravitational saddles w/ $AdS_2 \times S^2$ b.c (topologies, ~~saddle~~ orbifolds, ...)

§ Evaluation of Z_{AdS_2}

① $$\overset{H.D}{S_{eff}} = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\, R + G\int d^4x \sqrt{-g}\, R^2 + G^2 \int \dots$$

$$+ R_{ab}R^{ab} + R_{abcd}R^{abcd}$$

→ Higher-derivative terms arising from string compactifications ⇒ agree w/ microscopic calculations [cf Cardoso-de Wit-Kaeppeli-Mohaupt 2000]

② loops

• Fix fall-off conditions

e.g. $\phi(r,t) \underset{r\to\infty}{\sim} \phi_0(t) + \underbrace{\delta\varphi(r,t)}_{O(1/r^2)}$

A_μ —

$g_{\mu\nu}$ —

ψ —

. Then integrate over fluctuation.

→ Basis of eigenfunctions for Laplacian on AdS_2

$\varphi_k(r,t)$ $\qquad \delta\varphi(r,t) = \sum_k a_k \varphi_k(r,t)$

$$\int(\mathcal{D}\varphi) = \int \prod_k da_k \dots$$

$$\Rightarrow Z_{AdS_2}(\vec{\Phi},\vec{p}) \equiv e^{S^{qu}_{BH}(\vec{q},\vec{p}\,')}$$

$$= \exp\left(S^0_{BH}(\vec{q},\vec{p}\,') + C_{log}\log S^0_{BH} + \frac{C_1}{S^0_{BH}} + \frac{C_2}{(S^0_{BH})^2} + \dots \right)$$

$$+\, O(e^{-S^0_{BH}}) \dots$$

Results

- $C_{log} \to$ 1-loop effect, only sensitive to field content. [Sen + Banerjee + Gupta '10]

e.g. $\mathcal{N}=2$ ~~theory~~ sugra + n_V vectors + n_H hypers

$$C_{log} = \frac{1}{12}(23 + n_H - n_V).$$

- All-order perturbation theory using localization in sugra [Dabholkar - S.M. Gomes '10-'14]

- Nonperturbative effects

cf $\mathcal{N}=8$ BH

$$B_{14}(\Delta) = \sum_{c=1}^{\infty} \frac{1}{c^{9/2}} K_c(\Delta)\, \tilde{I}_{7/2}\left(\frac{\pi\sqrt{\Delta}}{c}\right)$$

$$= \tilde{I}_{7/2}(\pi\sqrt{\Delta})\,\left(\cancel{1 + O(e^{-\pi\sqrt{\Delta}/2}}\right.$$

localization result [DGM]

$$\exp\left(\pi\sqrt{\Delta} - 2\log\Delta + \cdots\right)$$

✓ area/4 C_{log} general ✓

$$+ \sum_{c=2}^{\infty} \boxed{\frac{1}{c}}\, K_c(\Delta)\, \frac{1}{c^{7/2}}\, \tilde{I}_{7/2}\left(\frac{\pi\sqrt{\Delta}}{c}\right)$$

orbifold $\dfrac{AdS_2 \times S^2 \times T^6}{\mathbb{Z}_c}$

Kloosterman sum from C.S theory (topological d.o.f on orbifold).

Result of same localization on orbifold

§ Zero modes in AdS_2

$r \to \infty \quad ds^2 \sim \frac{dr^2}{r^2} + r^2 dt^2$

- Path integral over modes normalizable on AdS_2.

e.g. $\int d^2x \sqrt{-g}\, (\partial_\mu \delta\varphi)^2$

$$= \underbrace{\int dt}_{2\pi} \int^{\infty} dr\, (\partial_r \delta\varphi)^2 r^2 < \infty .$$

$$\Rightarrow \partial_r \delta\varphi \sim \frac{1}{r^{2+\varepsilon}} \quad \varepsilon > 0 \quad \Rightarrow \quad \delta\varphi \sim \frac{1}{r^{1+\varepsilon}}$$

$$\delta\varphi = \overset{\times}{\varphi_0} + \overset{\times}{\frac{\varphi_1}{r}} + \overset{\checkmark}{\frac{\varphi_2}{r^2}}$$

Now, Fact: In Euclidean AdS_2 geometry, $\exists$ normalizable zero modes of Laplacian associated to any gauge symmetry.

Ref: Camporesi–Higuchi '94, Sen 1108.3842.

e.g. $A^0 = d\lambda$ (Maxwell $U(1)$ gauge field)
$\uparrow$ gauge parameter.

As $r \to \infty$, $\lambda \sim \lambda_0 + \frac{\lambda_1}{r} + \dots \Rightarrow A \sim \frac{\lambda_1}{r^2}$.

$\Rightarrow \lambda$ not normalizable, A normalizable.

$\Rightarrow$ Problem! $A^0 = d\lambda \Rightarrow F^0 = 0 \Rightarrow S_{\text{bulk}} = 0$ (also, $S_{\text{bdry}} = 0$)

$\Rightarrow \int DA^0 \cdot 1 \longrightarrow$ = Volume of space of zero modes $= \begin{cases} \infty ? \\ 0 ? \\ \text{finite?} \end{cases}$